

Interface dynamics of the porous medium equation with a bistable reaction term

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Abstract

We consider a degenerate partial differential equation arising in population dynamics, namely the porous medium equation with a bistable reaction term. We study its asymptotic behavior as a small parameter, related to the thickness of a diffuse interface, tends to zero. We prove the rapid formation of transition layers which then propagate. We prove the convergence to a sharp interface limit whose normal velocity, at each point, is that of the underlying degenerate travelling wave.

Key Words: degenerate diffusion, singular perturbation, sharp interface limit, population dynamics.²

1 Introduction

In this paper we consider the rescaled porous medium equation with a bistable reaction term

$$(P^\varepsilon) \quad \begin{cases} u_t = \varepsilon \Delta(u^m) + \frac{1}{\varepsilon} f(u) & \text{in } \Omega \times (0, \infty) \\ \frac{\partial(u^m)}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

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and study the sharp interface limit as $\varepsilon \rightarrow 0$. Here Ω is a smooth bounded domain in \mathbb{R}^N ($N \geq 2$), ν is the Euclidian unit normal vector exterior to $\partial\Omega$ and $m > 1$.

We assume that f is smooth, has exactly three zeros $0 < a < 1$ such that

$$f'(0) < 0, \quad f'(a) > 0, \quad f'(1) < 0, \quad (1.1)$$

and that

$$\int_0^1 m u^{m-1} f(u) du > 0. \quad (1.2)$$

The above assumption implies that the speed of the underlying degenerate travelling wave is positive (see subsection 3.1), so that the region enclosed by the limit interface is expanding (see below). This explains why the requirement (1.2) is convenient for the study of invasion processes.

As far as the initial data is concerned, we assume that $0 \leq u_0 \leq M$ (for some $M > a$) is a $C^2(\overline{\Omega})$ function with compact support

$$\text{Supp } u_0 := \text{Cl}\{x \in \Omega : u_0(x) > 0\} \subset\subset \Omega.$$

Furthermore we define the *initial interface* Γ_0 by

$$\Gamma_0 := \{x \in \Omega : u_0(x) = a\},$$

and suppose that Γ_0 is a smooth hypersurface without boundary, such that, n being the Euclidian unit normal vector exterior to Γ_0 ,

$$\Gamma_0 \subset\subset \Omega \quad \text{and} \quad \nabla u_0(x) \neq 0 \quad \text{if } x \in \Gamma_0, \quad (1.3)$$

$$u_0 > a \quad \text{in} \quad \Omega_0^{(1)}, \quad u_0 < a \quad \text{in} \quad \Omega_0^{(0)}, \quad (1.4)$$

where $\Omega_0^{(1)}$ denotes the region enclosed by Γ_0 and $\Omega_0^{(0)}$ the region enclosed between $\partial\Omega$ and Γ_0 .

Problem (P^ε) possesses a unique weak solution u^ε as it is explained in Section 2. As $\varepsilon \rightarrow 0$, by formally neglecting the diffusion term, we see that, in the very early stage, the value of u^ε quickly becomes close to either 1 or 0 in most part of Ω , creating a steep interface (transition layers) between the regions $\{u^\varepsilon \approx 1\}$ and $\{u^\varepsilon \approx 0\}$. Once such an interface develops, the diffusion term is large near the interface and comes to balance with the reaction term. As a result, the interface ceases rapid development and starts to propagate in a slower time scale. Therefore the limit solution $\tilde{u}(x, t)$ will be a step function taking the value 1 on one side of the moving interface, and 0 on the other side.

We shall prove that this sharp interface limit, which we denote by Γ_t , obeys the law of motion

$$(P^0) \quad \begin{cases} V_n = c^* & \text{on } \Gamma_t \\ \Gamma_t|_{t=0} = \Gamma_0, \end{cases}$$

where V_n is the normal velocity of Γ_t in the exterior direction, and c^* the positive speed of the underlying travelling wave (see subsection 3.1). Problem (P^0) possesses a unique smooth solution on $[0, T^{max})$ for some $T^{max} > 0$. We denote this solution by $\Gamma = \cup_{0 \leq t < T^{max}} (\Gamma_t \times \{t\})$. From now on, we fix $0 < T < T^{max}$ and work on $[0, T]$.

We set

$$Q_T := \Omega \times (0, T),$$

and, for each $t \in [0, T]$, we denote by $\Omega_t^{(1)}$ the region enclosed by the hypersurface Γ_t , and by $\Omega_t^{(0)}$ the region enclosed between $\partial\Omega$ and Γ_t . We define a step function $\tilde{u}(x, t)$ by

$$\tilde{u}(x, t) := \begin{cases} 1 & \text{in } \Omega_t^{(1)} \\ 0 & \text{in } \Omega_t^{(0)} \end{cases} \quad \text{for } t \in [0, T], \quad (1.5)$$

which represents the formal asymptotic limit of u^ε as $\varepsilon \rightarrow 0$.

Our main result, Theorem 1.1, describes both the emergence and the propagation of the layers. First, it gives the profile of the solution after a very short initial period: the solution u^ε quickly becomes close to 1 or 0, except in a small neighborhood of the initial interface Γ_0 , creating a steep transition layer around Γ_0 (*generation of interface*). The time needed to develop such a transition layer, which we will denote by t^ε , is $\mathcal{O}(\varepsilon |\ln \varepsilon|)$. Then the theorem states that the solution u^ε remains close to the step function \tilde{u} on the time interval $[t^\varepsilon, T]$ (*motion of interface*).

Theorem 1.1 (Generation and motion of interface). *Assume $m \geq 2$. Define μ as the derivative of $f(u)$ at the unstable equilibrium $u = a$, that is*

$$\mu = f'(a).$$

Let $\eta \in (0, \min(a, 1 - a))$ be arbitrary. Fix $\alpha_0 > 0$ arbitrarily small. Then there exist positive constants ε_0 and \mathcal{C} such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all (x, t) such that $t^\varepsilon \leq t \leq T$, where

$$t^\varepsilon := \mu^{-1} \varepsilon |\ln \varepsilon|, \quad (1.6)$$

we have

$$u^\varepsilon(x, t) \in \begin{cases} [1 - 2\varepsilon, 1 + \eta] & \text{if } x \in \Omega_t^{(1)} \setminus \mathcal{N}_{\mathcal{C}\varepsilon|\ln \varepsilon|}(\Gamma_t) \\ [0, 1 + \eta] & \text{if } x \in \Omega \\ [0, \eta] & \text{if } x \in \Omega_t^{(0)} \setminus \mathcal{N}_{\alpha_0}(\Gamma_t), \end{cases} \quad (1.7)$$

where $\mathcal{N}_r(\Gamma_t) := \{x \in \Omega : \text{dist}(x, \Gamma_t) < r\}$ denotes the r -tubular neighborhood of Γ_t .

Remark 1.2 (About the thickness of the interface). Since the construction of super-solutions is much more involved than that of sub-solutions, the statement (1.7) is more accurate in $\Omega_t^{(1)}$ than in $\Omega_t^{(0)}$. More precisely, on the one hand, (1.7) shows that the convergence to 1 is uniform “inside the interface” except in $\mathcal{O}(\varepsilon|\ln \varepsilon|)$ tubular neighborhoods of the sharp interface limit; on the other hand, (1.7) only shows that the convergence to 0 is uniform “outside the interface” except in $\mathcal{O}(1)$ tubular neighborhoods of the sharp interface limit.

Remark 1.3 (About the assumption $m \geq 2$). Note that the sub- and super-solutions we shall construct to study the motion of interface allow $m > 1$. Nevertheless, since we consider *not well-prepared initial data*, we need to quote a generation of interface result from [2] which is valid only for $m \geq 2$ (if $1 < m < 2$ the partial differential equation is not only degenerate but also singular). When initial data have a “suitable shape”, the restriction $m \geq 2$ can be removed.

As a direct consequence of Theorem 1.1, we have the following convergence result.

Corollary 1.4 (Convergence). *Assume $m \geq 2$. As $\varepsilon \rightarrow 0$, the solution u^ε converges to \tilde{u} in $\cup_{0 < t \leq T}(\Omega_t^{(i)} \times \{t\})$, where $i = 0, 1$.*

For the relevance of nonlinear diffusion in population dynamics models, we refer the reader to Gurney and Nisbet [9], Gurtin and Mac Camy [10]: density dependent diffusion is efficient to study the dynamics of a population which regulates its size below the carrying capacity set by the supply of nutrients. Since density dependent equations degenerate at points where $u = 0$, a loss of regularity of solutions occurs and their support propagates at finite speed.

Let us mention some earlier works on problems involving nonlinear diffusion that are related to ours. Feireisl [6] has studied the singular limit of (P^ε) in the whole space \mathbb{R}^N , which allows to reduce the issue to the radially symmetric case. Hilhorst, Kersner, Logak and Mimura [11] have investigated the singular limit of the equation posed in a bounded domain of \mathbb{R}^N , with $f(u)$ of the Fisher-KPP type. Note that the authors in [11] assume the convexity of $\Omega_0^{(1)}$ which allows them to construct a single super-solution for both the generation and the motion of interface. Here, we do not make such a geometric assumption.

The organization of this work is as follows. In Section 2, we briefly recall known results concerning the well-posedness of Problem (P^ε) . Section 3 is the body of the paper: we construct sub- and super-solutions to study the motion of the transition layers. Finally, we prove Theorem 1.1 in Section 4.

2 Comparison principle, well-posedness

Since the diffusion term degenerates when $u = 0$ a loss of regularity of solutions occurs. We define below a notion of weak solution for Problem (P^ε) , which is very similar to the one proposed by Aronson, Crandall and Peletier [3] for the one dimensional problem with homogeneous Dirichlet boundary conditions. Concerning the initial data, we suppose here that $u_0 \in L^\infty(\Omega)$ and $u_0 \geq 0$ a.e. Note that in this subsection, and only in this subsection, we assume, without loss of generality, that $\varepsilon = 1$, which yields the Problem

$$(P) \quad \begin{cases} u_t = \Delta(u^m) + f(u) & \text{in } \Omega \times (0, \infty) \\ \frac{\partial(u^m)}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Definition 2.1. A function $u : [0, \infty) \rightarrow L^1(\Omega)$ is a solution of Problem (P) if, for all $T > 0$,

(i) $u \in C([0, \infty); L^1(\Omega)) \cap L^\infty(Q_T)$;

(ii) for all $\varphi \in C^2(\overline{Q_T})$ such that $\varphi \geq 0$ and $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$, it holds that

$$\int_{\Omega} u(T)\varphi(T) - \int \int_{Q_T} (u\varphi_t + u^m \Delta \varphi) = \int_{\Omega} u_0 \varphi(0) + \int \int_{Q_T} f(u)\varphi. \quad (2.1)$$

A sub-solution (respectively a super-solution) of Problem (P) is a function satisfying (i) and (ii) with equality replaced by \leq (respectively \geq).

Theorem 2.2 (Existence and comparison principle). *The following properties hold.*

(i) Let u^- and u^+ be a sub-solution and a super-solution of Problem (P^ε) with initial data u_0^- and u_0^+ respectively.

$$\text{If } u_0^- \leq u_0^+ \text{ a.e. then } u^- \leq u^+ \text{ in } Q_T;$$

(ii) Problem (P) has a unique solution u on $[0, \infty)$ and

$$0 \leq u \leq \max(1, \|u_0\|_{L^\infty(\Omega)}) \text{ in } Q_T; \quad (2.2)$$

(iii) $u \in C(\overline{Q_T})$.

The proof of the theorem above can be performed in the same lines as in [3, Theorem 5] (see also [13] and [4] for related results). The continuity of u^ε follows from [5].

The following result turns out to be an essential tool when constructing smooth sub- and super-solutions of Problem (P^ε) .

Lemma 2.3. *Let u be a continuous nonnegative function in $\overline{\Omega} \times [0, T]$. Define $\Omega_t^* = \{x \in \Omega : u(x, t) > 0\}$ and $\Gamma_t^* = \partial\Omega_t^*$ for all $t \in [0, T]$. Suppose the family $\Gamma := \cup_{0 < t \leq T} \Gamma_t^* \times \{t\}$ is sufficiently smooth and let ν_t^* be the outward normal vector on Γ_t^* . Suppose moreover that*

- (i) $\nabla(u^m)$ is continuous in $\overline{\Omega} \times [0, T]$
- (ii) $\mathcal{L}[u] := u_t - \Delta(u^m) - f(u) = 0$ in $\{(x, t) \in \overline{\Omega} \times [0, T] : u(x, t) > 0\}$
- (iii) $\frac{\partial(u^m)}{\partial\nu_t^*} = 0$ on $\partial\Omega_t^*$, for all $t \in [0, T]$.

Then u is a solution of Problem (P). Similarly a sub-solution (respectively a super-solution) of Problem (P) is a function satisfying (i) and (ii)—(iii) with equality replaced by \leq (respectively \geq).

The proof of this result can be found in [11].

3 Motion of the transition layers

3.1 Materials

Underlying travelling waves. Hosono [12] has investigated travelling wave solutions for the degenerate one dimensional equation

$$u_t = (u^m)_{xx} + f(u).$$

He proved that there exists a unique travelling wave (c^*, U) , that the sign of the velocity c^* is that of $\int_0^1 u^{m-1} f(u) du$, and that the profiles vary with the sign of the velocity. More precisely, for $c^* < 0$, the front is smooth and $U \in C^\infty(\mathbb{R})$, whereas, for $c^* > 0$, we only have $(U^{m-1})' \in L^\infty(\mathbb{R})$, but $(U^{m-1})' \notin C(\mathbb{R})$. These different behaviors of the travelling waves are in contrast with the density independent diffusion models, where fronts are smooth whatever their velocities are (see [7]).

In the present paper, the assumption (1.2) implies that $c^* > 0$. More precisely the following holds (see [12] for details). The travelling wave (c^*, U) is the solution of the auxiliary problem

$$\left\{ \begin{array}{ll} (U^m)'' + c^* U' + f(U) = 0 & \text{on } (-\infty, \omega) \\ U(-\infty) = 1 \\ U(0) = a \\ U' < 0 & \text{on } (-\infty, \omega) \\ (U^m)'(\omega) = 0 \\ U \equiv 0 & \text{on } [\omega, \infty), \end{array} \right. \quad (3.1)$$

for some $\omega > 0$. As $z \rightarrow -\infty$, terms are exponentially decaying:

$$\max(1 - U(z), |U'(z)|, |U''(z)|) \leq Ce^{-\lambda|z|} \quad \text{for } z \leq 0, \quad (3.2)$$

for some positive constants C and λ . As $z \nearrow \omega$, we have

$$\lim_{z \nearrow \omega} (U^{m-1})'(z) = -\frac{m-1}{m}c^* \quad \text{and} \quad \lim_{z \nearrow \omega} (U^{m-1})''(z) = -\frac{(m-1)^2}{m^2}f'(0), \quad (3.3)$$

and $U'(\omega) \in [-\infty, 0)$. Moreover, for a positive constant which we denote again by C , there holds

$$|(U^m)''(z)| \leq C|U'(z)| = -CU'(z) \quad \text{for all } z \in (-\infty, \omega). \quad (3.4)$$

The cut-off signed distance function. Another classical ingredient in similar situations (see [14] or [8]) is a *cut-off signed distance function* d which we now define. Let $\tilde{d}(\cdot, t)$ be the signed distance function to Γ_t , namely

$$\tilde{d}(x, t) := \begin{cases} -\text{dist}(x, \Gamma_t) & \text{for } x \in \Omega_t^{(1)} \\ \text{dist}(x, \Gamma_t) & \text{for } x \in \Omega_t^{(0)}, \end{cases} \quad (3.5)$$

where $\text{dist}(x, \Gamma_t)$ is the distance from x to the hypersurface Γ_t . We remark that $\tilde{d}(\cdot, t) = 0$ on Γ_t and that $|\nabla \tilde{d}| = 1$ in a neighborhood of the interface, say $|\nabla \tilde{d}(x, t)| = 1$ if $|\tilde{d}(x, t)| \leq 2d_0$, for some $d_0 > 0$. By reducing d_0 if necessary we can assume that \tilde{d} is smooth in $\{(x, t) \in \overline{\Omega} \times [0, T] : |\tilde{d}(x, t)| < 3d_0\}$ and that

$$\text{dist}(\Gamma_t, \partial\Omega) \geq 3d_0 \quad \text{for all } t \in [0, T]. \quad (3.6)$$

Next, let $\zeta(s)$ be a smooth increasing function on \mathbb{R} such that

$$\zeta(s) = \begin{cases} s & \text{if } |s| \leq d_0 \\ -2d_0 & \text{if } s \leq -2d_0 \\ 2d_0 & \text{if } s \geq 2d_0. \end{cases}$$

We then define the cut-off signed distance function d by

$$d(x, t) := \zeta(\tilde{d}(x, t)). \quad (3.7)$$

Note that

$$\text{if } |d(x, t)| < d_0 \quad \text{then} \quad |\nabla d(x, t)| = 1, \quad (3.8)$$

that d is constant ($= 2d_0$) in a neighborhood of $\partial\Omega$, and that the equation of motion (P^0) yields

$$\text{if } |d(x, t)| < d_0 \quad \text{then} \quad d_t(x, t) + c^* = 0. \quad (3.9)$$

Moreover, there exists a constant $C > 0$ such that

$$|\nabla d(x, t)| + |\Delta d(x, t)| \leq C \quad \text{for all } (x, t) \in \overline{Q_T}. \quad (3.10)$$

3.2 Construction of sub-solutions

Equipped with the travelling wave (c^*, U) and the signed distance function d , we are looking for sub-solutions in the form

$$u_\varepsilon^-(x, t) := (1 - \varepsilon)U\left(\frac{d(x, t) + \varepsilon|\ln \varepsilon|p e^t}{\varepsilon}\right) = (1 - \varepsilon)U(z_\varepsilon^-(x, t)), \quad (3.11)$$

where

$$z_\varepsilon^-(x, t) := \frac{d(x, t) + \varepsilon|\ln \varepsilon|p e^t}{\varepsilon}. \quad (3.12)$$

Lemma 3.1 (Sub-solutions). *Let $p > 0$ be arbitrary. Then, for $\varepsilon > 0$ small enough, u_ε^- is a sub-solution for Problem (P^ε) .*

Proof. In this proof (and only in this proof) we set $u_\varepsilon^- = u$ and $z_\varepsilon^- = z$. Note that

$$\Omega_t^* = \{x \in \Omega : d(x, t) < -\varepsilon|\ln \varepsilon|p e^t + \varepsilon\omega\}, \quad (3.13)$$

where Ω_t^* is defined as in Lemma 2.3. It follows that $u \equiv 0$ near the boundary $\partial\Omega$ so that the Neumann boundary condition (iii) in Lemma 2.3 is fulfilled. Since $(U^m)'(\omega) = 0$ we see that $\nabla(u^m)$ is continuous in $\overline{\Omega} \times [0, T]$. Therefore, by virtue of Lemma 2.3, it is enough to prove that

$$\varepsilon \mathcal{L}^\varepsilon[u] := \varepsilon u_t - \varepsilon^2 \Delta(u^m) - f(u) \leq 0$$

in $\{(x, t) : d(x, t) < -\varepsilon|\ln \varepsilon|p e^t + \varepsilon\omega\} = \{(x, t) : z(x, t) < \omega\}$.

By using straightforward computations we get

$$\begin{aligned} \varepsilon u_t &= (1 - \varepsilon) (d_t + \varepsilon|\ln \varepsilon|p e^t) U'(z) \\ \varepsilon^2 \Delta(u^m) &= (1 - \varepsilon)^m |\nabla d|^2 (U^m)''(z) + (1 - \varepsilon)^m \varepsilon \Delta d (U^m)'(z), \end{aligned}$$

where $z = z(x, t)$. Then using the ordinary differential equation $(U^m)'' + c^* U' + f(U) = 0$, we see that

$$\varepsilon \mathcal{L}^\varepsilon[u] = E_1 + \cdots + E_4,$$

with

$$E_1 := (1 - \varepsilon) [d_t + c^* - (1 - \varepsilon)^{m-1} \varepsilon \Delta d (m U^{m-1})'(z) + \varepsilon|\ln \varepsilon|p e^t] U'(z)$$

$$E_2 := (1 - \varepsilon)^m (1 - |\nabla d|^2) (U^m)''(z)$$

$$E_3 := ((1 - \varepsilon) - (1 - \varepsilon)^m) (U^m)''(z)$$

$$E_4 := -f((1 - \varepsilon)U(z)) + (1 - \varepsilon)f(U(z)).$$

In the following we shall denote by C some positive constants which do not depend on $\varepsilon > 0$ small enough (and may change from place to place).

We start with some observations on the term E_4 . Note that

$$f((1-\varepsilon)u) - (1-\varepsilon)f(u) = -\varepsilon u f'(\theta) + \varepsilon f(u), \quad (3.14)$$

for some $\theta \in ((1-\varepsilon)u, u)$. Hence

$$|E_4| \leq C\varepsilon.$$

Moreover, since $f(1) = 0$ and $f'(1) < 0$, it follows from (3.14) that, for u sufficiently close to 1,

$$f((1-\varepsilon)u) - (1-\varepsilon)f(u) \geq \beta\varepsilon u, \quad (3.15)$$

for some $\beta > 0$. Hence since $U(-\infty) = 1$, by choosing $\gamma \gg 1$ we see that

$$E_4 \leq -\beta\varepsilon U(z) \leq -\frac{1}{2}\beta\varepsilon \quad \text{for all } z \leq -\gamma. \quad (3.16)$$

In the following we distinguish three cases, namely (3.17), (3.19) and (3.20).

Assume that

$$-\varepsilon|\ln \varepsilon|pe^t - \varepsilon\gamma \leq d(x, t) < -\varepsilon|\ln \varepsilon|pe^t + \varepsilon\omega, \quad (3.17)$$

which in turn implies that $-\gamma \leq z < \omega$. Since $U' < 0$ on $(-\infty, \omega)$ and $U'(\omega) \in [-\infty, 0)$, it holds that $U'(z) \leq -\alpha$, for some $\alpha > 0$. If $\varepsilon > 0$ is small enough (3.8) shows that $E_2 = 0$; from (3.4) we deduce that $|E_3| \leq -C\varepsilon U'(z)$; moreover we have $|E_4| \leq C\varepsilon$. In view of (3.9), E_1 reduces to

$$E_1 = (1-\varepsilon) \left[-(1-\varepsilon)^{m-1} \varepsilon \Delta d(mU^{m-1})(z) + \varepsilon|\ln \varepsilon|pe^t \right] U'(z). \quad (3.18)$$

Since $|-(1-\varepsilon)^{m-1} \varepsilon \Delta d(mU^{m-1})(z)| \leq C\varepsilon$, inequality $E_1 \leq \frac{1}{2}p\varepsilon|\ln \varepsilon|U'(z)$ holds. Collecting theses estimates we have

$$\begin{aligned} \mathcal{L}^\varepsilon[u] &\leq (\tfrac{1}{2}p\varepsilon|\ln \varepsilon| - C\varepsilon)U'(z) + C\varepsilon \\ &\leq -\tfrac{1}{4}p\alpha\varepsilon|\ln \varepsilon| + C\varepsilon \\ &\leq 0, \end{aligned}$$

if $\varepsilon > 0$ is sufficiently small.

Assume that

$$-d_0 \leq d(x, t) < -\varepsilon|\ln \varepsilon|pe^t - \varepsilon\gamma, \quad (3.19)$$

which in turn implies that $z < -\gamma$ so that (3.16) implies $E_4 \leq 0$. Here again (3.8) shows that $E_2 = 0$, from (3.4) we deduce that $|E_3| \leq -C\varepsilon U'(z)$, and E_1 reduces to (3.18). Hence we collect

$$\mathcal{L}^\varepsilon[u] \leq U'(z) [-(1-\varepsilon)\varepsilon C + (1-\varepsilon)p\varepsilon|\ln \varepsilon| - C\varepsilon] \leq 0,$$

for $\varepsilon > 0$ small enough.

Assume that

$$-2d_0 \leq d(x, t) < -d_0, \quad (3.20)$$

which in turn implies that, for $\varepsilon > 0$ small enough, $z \leq -\frac{d_0}{2\varepsilon}$. In this range (3.8) and (3.9) no longer apply but the exponential decay (3.2) shows that $|E_1| + |E_2| + |E_3| \leq Ce^{-\lambda\frac{d_0}{2\varepsilon}}$. Last $E_4 \leq -\frac{1}{2}\beta\varepsilon$ (see (3.16)) shows that, for $\varepsilon > 0$ small enough, $\mathcal{L}^\varepsilon[u] \leq 0$.

The lemma is proved. \square

3.3 Construction of super-solutions

The construction of super-solutions is more involved: since we want them to be positive it is no longer possible to use the natural travelling wave (c^*, U) which is compactly supported. Therefore we shall first consider slightly larger speeds $c > c^*$ which provide faster travelling wave solutions which tend to $+\infty$ in $-\infty$; then a small modification will provide us positive and more regular functions which are “nearly” travelling wave solutions. Before making this argument more precise, let us note that, as it will clearly appear below, the possibility of the above strategy follows from [12].

Let $\eta \in (0, \min(a, 1 - a))$ be arbitrary. Let $\alpha_0 > 0$ be fixed. Let us recall that we have fixed $0 < T < T^{max}$, where T^{max} denotes the time when the first singularities occur in (P^0) . Therefore we can select $\rho > 0$ small enough so that the following holds. First the smooth solution (Γ_t^c) of the free boundary problem

$$(P_c^0) \quad \begin{cases} V_n = c := c^* + \rho & \text{on } \Gamma_t^c \\ \Gamma_t^c|_{t=0} = \Gamma_0, \end{cases}$$

exists at least on $[0, T]$. Secondly, if we denote by $d^c(x, t)$ the cut-off signed distance function associated with $\Gamma^c := \cup_{0 \leq t \leq T} (\Gamma_t^c \times \{t\})$ then, for all $(x, t) \in Q_T$,

$$d(x, t) \geq \alpha_0 \implies d^c(x, t) \geq \frac{\alpha_0}{2}. \quad (3.21)$$

Since $c > c^*$, as explained in [12, Remark 3.1], there exists a faster travelling wave (c, V) which satisfies the same requirements as (c^*, U) in the auxiliary problem (3.1), except that $V(-\infty) = +\infty$ rather than $U(-\infty) = 1$. In particular, V is still compactly supported from one side.

Next, for all $n \geq 1$, following the construction which comes before Proposition 4.1 in [12] (it consists in slightly modifying the above travelling wave (c, V)), we can consider (c, U_n) such that

(i) U_n satisfies the ordinary differential equation

$$(U_n^m)'' + cU_n' + f(U_n) = 0 \quad \text{on some } (-\infty, Z_n),$$

where $U_n' < 0$ holds

(ii) U_n is constant equal to some $(\delta_n)^{\frac{1}{m-1}} > 0$ on $[Z_n, \infty)$

(iii) $(U_n)^{m-1} \in C^1(\mathbb{R})$

together with $U_n(0) = a$ and $U_n(-\infty) = +\infty$. Moreover $\delta_n \rightarrow 0$ as $n \rightarrow \infty$, so that we can fix $n_0 \gg 1$ such that $(\delta_{n_0})^{\frac{1}{m-1}} \leq \eta$.

As a conclusion, if we denote U_{n_0} , δ_{n_0} and Z_{n_0} by W , δ and Z we are now equipped with (c, W) such that $W^{m-1} \in C^1(\mathbb{R})$ and

$$\begin{cases} (W^m)'' + cW' + f(W) = 0 & \text{on } (-\infty, Z) \\ W(-\infty) = +\infty \\ W(0) = a \\ W' < 0 & \text{on } (-\infty, Z) \\ W \equiv \delta^{\frac{1}{m-1}} \leq \eta & \text{on } [Z, \infty). \end{cases} \quad (3.22)$$

We are now looking for super-solutions in the form

$$u_\varepsilon^+(x, t) := W\left(\frac{d^c(x, t) - \varepsilon|\ln \varepsilon|Ke^t}{\varepsilon}\right). \quad (3.23)$$

In the sequel we set

$$z_\varepsilon^+(x, t) := \frac{d^c(x, t) - \varepsilon|\ln \varepsilon|Ke^t}{\varepsilon}. \quad (3.24)$$

Remark 3.2 (The sub-domain Σ). We shall consider below a sub-domain Σ whose slice at time t , namely $\sigma_t := \{x : (x, t) \in \Sigma\}$, is the region enclosed between $\partial\Omega$ and (more or less) Γ_t^c . We shall prove that u_ε^+ is a super-solution in Σ . Thanks to (4.4) this will be sufficient for our purpose (see Section 4).

Denote by $-\theta$ the point where $W(-\theta) = 1 + \eta$. For each $0 \leq t \leq T$, define the open set

$$\sigma_t := \{x \in \Omega : d^c(x, t) > \varepsilon|\ln \varepsilon|Ke^t - \varepsilon\theta\} = \{x : z_\varepsilon^+(x, t) > -\theta\},$$

and the sub-domain

$$\Sigma := \cup_{0 < t < T} (\sigma_t \times \{t\}).$$

Note that the lateral boundary of Σ is made of $\partial_{\text{out}}\Sigma := \partial\Omega \times (0, T)$ and $\partial_{\text{in}}\Sigma := \cup_{0 < t < T} (s_t \times \{t\})$ where s_t denotes the smooth hypersurface

$$s_t := \{x \in \Omega : d^c(x, t) = \varepsilon|\ln \varepsilon|Ke^t - \varepsilon\theta\} = \{x : z_\varepsilon^+(x, t) = -\theta\}.$$

Lemma 3.3 (Super-solutions in Σ). *Let $\eta \in (0, \min(a, 1 - a))$ be arbitrary and let $\alpha_0 > 0$ be fixed. Then, for all $K > 0$, all $\varepsilon > 0$ small enough, u_ε^+ is such that*

- (i) $\mathcal{L}^\varepsilon[u_\varepsilon^+] := (u_\varepsilon^+)_t - \varepsilon \Delta((u_\varepsilon^+)^m) - \frac{1}{\varepsilon} f(u_\varepsilon^+) \geq 0$ in Σ
- (ii) $\frac{\partial((u_\varepsilon^+)^m)}{\partial \nu} = 0$ on $\partial_{out}\Sigma = \partial\Omega \times (0, T)$
- (iii) $u_\varepsilon^+ \equiv 1 + \eta$ on $\partial_{in}\Sigma = \cup_{0 < t < T}(s_t \times \{t\})$.

Proof. In this proof (and only in this proof) we put $u_\varepsilon^+ = u$ and $z_\varepsilon^+ = z$. Recall that d^c is constant near the boundary $\partial\Omega$ so that the Neumann boundary condition (ii) is fulfilled. Moreover the Dirichlet boundary condition (iii) is clear from the definition of s_t and the fact that $W(-\theta) = 1 + \eta$.

Therefore it remains to prove that $\varepsilon \mathcal{L}^\varepsilon[u] = \varepsilon u_t - \varepsilon^2 \Delta(u^m) - f(u) \geq 0$ in $\Sigma = \{(x, t) : z(x, t) > -\theta\}$. If $z(x, t) \geq Z$ then $\mathcal{L}^\varepsilon[u] = \mathcal{L}^\varepsilon[\delta^{\frac{1}{m-1}}] \geq 0$. We now assume that $z(x, t) \in (-\theta, Z)$, i.e.

$$\varepsilon |\ln \varepsilon| K e^t - \varepsilon \theta < d^c(x, t) < \varepsilon |\ln \varepsilon| K e^t + \varepsilon Z. \quad (3.25)$$

Straightforward computations combined with the ordinary differential equation $(W^m)'' + cW' + f(W) = 0$ yield

$$\varepsilon \mathcal{L}^\varepsilon[u] = (d_t^c + c)W' - \varepsilon |\ln \varepsilon| K e^t W' - \varepsilon \Delta d^c(W^m)' + (1 - |\nabla d^c|^2)(W^m)''.$$

If $\varepsilon > 0$ is small enough, then (3.25) combined with (3.8) and (3.9) — with d^c and c playing the roles of d and c^* — shows that the above equality reduces to

$$\varepsilon \mathcal{L}^\varepsilon[u] = -\varepsilon W' (|\ln \varepsilon| K e^t + \Delta d^c(mW^{m-1})).$$

Since $W' \leq 0$ we have $\varepsilon \mathcal{L}^\varepsilon[u] \geq 0$, for $\varepsilon > 0$ small enough.

The lemma is proved. \square

4 Proof of Theorem 1.1

4.1 A generation of interface property

We first state a result on the generation of interface.

Lemma 4.1 (Generation of interface). *Assume $m \geq 2$. Let $\eta > 0$ be arbitrary small. Then, for all $x \in \Omega$, we have, for $\varepsilon > 0$ small enough,*

$$0 \leq u^\varepsilon(x, t^\varepsilon) \leq 1 + \eta, \quad (4.1)$$

and there exists $M_0 > 0$ such that, for $\varepsilon > 0$ small enough,

$$u_0(x) \geq a + M_0 \varepsilon |\ln \varepsilon| \implies u^\varepsilon(x, t^\varepsilon) \geq 1 - \varepsilon \quad (4.2)$$

$$u_0(x) \leq a - M_0 \varepsilon |\ln \varepsilon| \implies u^\varepsilon(x, t^\varepsilon) \leq \varepsilon, \quad (4.3)$$

where $t^\varepsilon = \mu^{-1} \varepsilon |\ln \varepsilon|$.

Proof. We only give an outline since the arguments can be found in [1], [2].

Denote by $Y(\tau; \xi)$ the solution of the bistable ordinary differential equation $Y_\tau = f(Y)$ on $(0, \infty)$ supplemented with the initial condition $Y(0; \xi) = \xi$. Modulo a change of the time variable, we can use the sub- and super-solutions constructed in [2] to deduce that, for some $C^* > 0$, for $\varepsilon > 0$ small enough,

$$\begin{aligned} & \left[Y\left(\frac{t^\varepsilon}{\varepsilon}; u_0(x) - \varepsilon^2 C^* (e^{\mu t^\varepsilon/\varepsilon} - 1)\right) \right]^+ \\ & \leq u^\varepsilon(x, t^\varepsilon) \leq \left[Y\left(\frac{t^\varepsilon}{\varepsilon}; u_0(x) + \varepsilon^2 C^* (e^{\mu t^\varepsilon/\varepsilon} - 1)\right) \right]^+ . \end{aligned}$$

Next a straightforward modification of [1, Lemma 3.9] —which specifies the instability of the equilibrium $Y \equiv a$ and the stability of the equilibria $Y \equiv 0$, $Y \equiv 1$ — shows that for $\varepsilon > 0$ small enough, for all $\xi \in (-1, 2)$, we have

$$Y(\mu^{-1}|\ln \varepsilon|; \xi) \leq 1 + \eta ,$$

and that

$$\begin{aligned} \xi \geq a + \varepsilon|\ln \varepsilon| &\implies Y(\mu^{-1}|\ln \varepsilon|; \xi) \geq 1 - \varepsilon \\ \xi \leq a - \varepsilon|\ln \varepsilon| &\implies Y(\mu^{-1}|\ln \varepsilon|; \xi) \leq \varepsilon . \end{aligned}$$

The combination of the above arguments completes the proof of the lemma. \square

4.2 Proof of the main theorem

We are now in the position to prove Theorem 1.1. To that purpose we show that solutions are in between the propagation of interface sub- and super-solutions at time t^ε .

Proof. Assume $m \geq 2$. Let $\eta \in (0, \min(a, 1 - a))$ be arbitrary. First note that the comparison principle directly implies that inequality (4.1) persists for later times, i.e.

$$u^\varepsilon(x, t) \in [0, 1 + \eta] \quad \text{for all } x \in \Omega \text{ and all } t^\varepsilon \leq t \leq T. \quad (4.4)$$

Since $\nabla u_0 \neq 0$ everywhere on $\Gamma_0 = \{x \in \Omega : u_0(x) = a\}$ and since Γ_0 is a compact hypersurface, we can find a positive constant M_1 such that

$$\begin{aligned} \text{if } d(x, 0) \leq -M_1 \varepsilon |\ln \varepsilon| & \quad \text{then } u_0(x) \geq a + M_0 \varepsilon |\ln \varepsilon| \\ \text{if } d(x, 0) \geq M_1 \varepsilon |\ln \varepsilon| & \quad \text{then } u_0(x) \leq a - M_0 \varepsilon |\ln \varepsilon| , \end{aligned} \quad (4.5)$$

with M_0 the constant which appears in Lemma 4.1.

We first investigate the behavior “inside the interface”. In view of (3.13), we can choose $p > 0$ large enough so that, for $\varepsilon > 0$ small enough, the sub-solution u_ε^- defined in (3.11) is such that the set $\{x : u_\varepsilon^-(x, 0) > 0\}$ is included in $\{x : d(x, 0) \leq -M_1\varepsilon|\ln \varepsilon|\}$. Therefore, from the correspondence (4.5), the estimate (4.2) and the fact that $u_\varepsilon^-(x, 0) \leq 1 - \varepsilon$, we deduce that, for all $x \in \Omega$,

$$u_\varepsilon^-(x, 0) \leq u^\varepsilon(x, t^\varepsilon).$$

It follows from the comparison principle that

$$u_\varepsilon^-(x, t - t^\varepsilon) = (1 - \varepsilon)U\left(\frac{d(x, t - t^\varepsilon) + \varepsilon|\ln \varepsilon|pe^{t-t^\varepsilon}}{\varepsilon}\right) \leq u^\varepsilon(x, t), \quad (4.6)$$

for all $(x, t) \in \Omega \times [t^\varepsilon, T]$. We choose $\mathcal{C} \gg \max(c^*\mu^{-1}, pe^T, \lambda)$ so that

$$\frac{-\mathcal{C}\varepsilon|\ln \varepsilon| + c^*\mu^{-1}\varepsilon|\ln \varepsilon| + \varepsilon|\ln \varepsilon|pe^{t-t^\varepsilon}}{\varepsilon} \leq -\frac{\mathcal{C}}{2}|\ln \varepsilon| \leq -\frac{2}{\lambda}|\ln \varepsilon|, \quad (4.7)$$

where $\lambda > 0$ is the constant appearing in (3.2). We take $x \in \Omega_t^{(1)} \setminus \mathcal{N}_{\mathcal{C}\varepsilon|\ln \varepsilon|}(\Gamma_t)$, i.e.

$$d(x, t) \leq -\mathcal{C}\varepsilon|\ln \varepsilon|, \quad (4.8)$$

and prove below that $u^\varepsilon(x, t) \geq 1 - 2\varepsilon$, for $t^\varepsilon \leq t \leq T$. Note that, for $\varepsilon > 0$ small enough,

$$d(x, t - t^\varepsilon) = d(x, t) + c^*t^\varepsilon, \quad (4.9)$$

which, combined with (4.6) and (4.7), implies

$$\begin{aligned} u^\varepsilon(x, t) &\geq (1 - \varepsilon)U\left(-\frac{2}{\lambda}|\ln \varepsilon|\right) \\ &\geq (1 - \varepsilon)(1 - C\varepsilon^2) \\ &\geq 1 - 2\varepsilon, \end{aligned}$$

where we have used (3.2).

Last we investigate the behavior “outside the interface”. Fix $\alpha_0 > 0$ arbitrarily small. For such $\alpha_0 > 0$, we follow the strategy of subsection 3.3 to construct super-solutions in Σ . More precisely define $K = 2M_1$, where $M_1 > 0$ is the constant that appears in (4.5); then construct $u_\varepsilon^+(x, t)$ as in (3.23). In view of Lemma 3.3 (ii), (iii), the super-solution $u_\varepsilon^+(x, t)$ and the solution $u^\varepsilon(x, t + t^\varepsilon)$ satisfy a Neumann boundary condition on $\partial_{\text{out}}\Sigma$ and —taking advantage of (4.4)— are ordered on $\partial_{\text{in}}\Sigma$. Last we claim that (note that $d^c(x, 0) = d(x, 0)$ since $\Gamma_0^c = \Gamma_0$)

$$u^\varepsilon(x, t^\varepsilon) \leq W\left(\frac{d(x, 0) - 2M_1\varepsilon|\ln \varepsilon|}{\varepsilon}\right) = u_\varepsilon^+(x, 0), \quad (4.10)$$

for all $x \in \sigma_0 = \{x : d(x, 0) > 2M_1\varepsilon|\ln \varepsilon| - \varepsilon\theta\}$. Indeed, (4.5) and (4.3) show that $u^\varepsilon(x, t^\varepsilon) \leq \varepsilon$ and the conclusion (4.10) follows from the fact that $\delta^{\frac{1}{m-1}} \leq W$. Hence the comparison principle yields

$$u^\varepsilon(x, t) \leq W \left(\frac{d^c(x, t - t^\varepsilon) - \varepsilon|\ln \varepsilon| 2M_1 e^{t-t^\varepsilon}}{\varepsilon} \right) = u_\varepsilon^+(x, t - t^\varepsilon), \quad (4.11)$$

for all $(x, t) \in \Sigma$ with $t^\varepsilon \leq t \leq T$. We take $x \in \Omega_t^{(0)} \setminus \mathcal{N}_{\alpha_0}(\Gamma_t)$, i.e.

$$d(x, t) \geq \alpha_0, \quad (4.12)$$

and prove below that $u^\varepsilon(x, t) \leq \eta$, for $t^\varepsilon \leq t \leq T$. From (3.21) we deduce that $(x, t) \in \Sigma$ so that (4.11) applies. Note also that $d^c(x, t - t^\varepsilon) = d^c(x, t) + ct^\varepsilon$. Therefore we infer from (3.21) that, for $\varepsilon > 0$ small enough, $d^c(x, t - t^\varepsilon) \geq \frac{\alpha_0}{3}$ which in turn implies

$$u_\varepsilon^+(x, t - t^\varepsilon) \leq W \left(\frac{\frac{\alpha_0}{3} - 2M_1 e^T \varepsilon |\ln \varepsilon|}{\varepsilon} \right) = \delta^{\frac{1}{m-1}} \leq \eta,$$

since $W(+\infty) = \delta^{\frac{1}{m-1}}$. Conclusion follows from (4.11).

Theorem 1.1 is proved. \square

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